Indian Institute of Information Technology Allahabad
End Sem Question paper
(Tentative Marking Scheme)

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Course Code: IT1005   Program Name: B.Tech. (IT & IT-Bin)
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1. Prove or disprove the following statements with proper justification. [4]

   (a) Let G be a simple graph with more than 3 vertices such that for each vertex v, $\deg(v) \geq \frac{n}{2}$. Then G is connected.

   Solution. By Dirac's theorem, G has a Hamiltonian cycle. Hence G is connected. [2]

   (b) The set $S = \left\{ \begin{pmatrix} a & \bar{b} \\ -b & \bar{a} \end{pmatrix} \mid a, b \in \mathbb{C} \right\}$ is a field under matrix addition and matrix multiplication.

   Solution. The set S under matrix multiplication is not commutative, hence is not a field. Let, $A = \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$ and $B = \begin{pmatrix} i & 1 \\ -1 & -i \end{pmatrix}$. Then $\begin{pmatrix} 0 & 0 \\ 0 & -2i \end{pmatrix} = AB \neq BA = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ [2]

2. Using generating function, find the number of non-negative integer solutions of $a+b+c+d+e = 27$, satisfying $3 \leq a, b, c \leq 8$, and others ≥ 0. [6]

   Solution The generating function for a, b, c each is $x^3 + x^4 + \cdots + x^8 = x^3(\frac{1-x^6}{1-x})$. [1]

   The generating function for d, e each is $x^0 + x^1 + x^2 + \cdots = \frac{1}{1-x}$. [1]

   Thus the generating function of the problem is

$$G(x) = x^9 \frac{(1-x^6)^3}{(1-x)^5} = \frac{x^9}{(1-x)^5} - 3\frac{x^{15}}{(1-x)^5} + 3\frac{x^{21}}{(1-x)^5} - \frac{x^{27}}{(1-x)^5}.$$

   Thus

$$[x^{27}]G(x) = [x^{18}]\frac{1}{(1-x)^5} - 3[x^{12}]\frac{1}{(1-x)^5} + 3[x^6]\frac{1}{(1-x)^5} - [x^0]\frac{1}{(1-x)^5}$$

$$= \binom{22}{4} - 3\binom{16}{4} + 3\binom{10}{4} - \binom{4}{4} = 2484.$$ [1 × 4]

3. A population evolves according to the following rule: at the end of each hour, the population becomes the sum of its population in the previous hour and twice its population two hours earlier, and then 3 new individuals are added. Initially (at 0 hours), the population is 2, and after 1 hour, the population is 5. [6]

   (a) Form a recurrence relation to model the population after n hours.
   Solution. Let a_n denote the population at n hours. Then the recurrence relation is

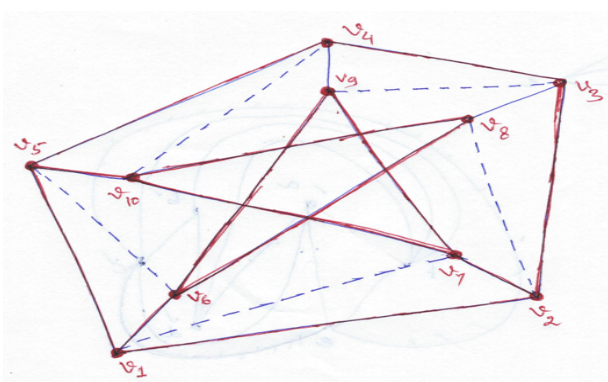
   $$a_n = a_{n-1} + 2a_{n-2} + 3, \quad a_0 = 2, \quad a_1 = 5. \qquad [2]$$

   (b) Solve the recurrence relation to obtain the population after n hours.
   Solution. The characteristic equation of the relation is $r^2 - r - 2 = 0$. This gives $r = 2, -1$. Hence $a_n^{(h)} = c_1 2^n + c_2(-1)^n$. [1]
   Note that $F(n) = 3.1^n$. So a particular solution is $a_n^{(p)} = k.1^n = k$. This gives $k = -\frac{3}{2}$. [1]
   Thus the general solutions is $a_n = c_1 2^n + c_2(-1)^n - \frac{3}{2}$.
   From the initial conditions, $c_1 = \frac{10}{3}$ and $c_2 = \frac{1}{6}$. [$2 \times \frac{1}{2}$]
   So, the population after n hours is $a_n = \frac{10}{3}.2^n + \frac{1}{6}.(-1)^n - \frac{3}{2}$.

   (c) What is the population after 10 hours?
   Solution. $a_{10} = 5120 - \frac{4}{3} \approx 5118$ [1]

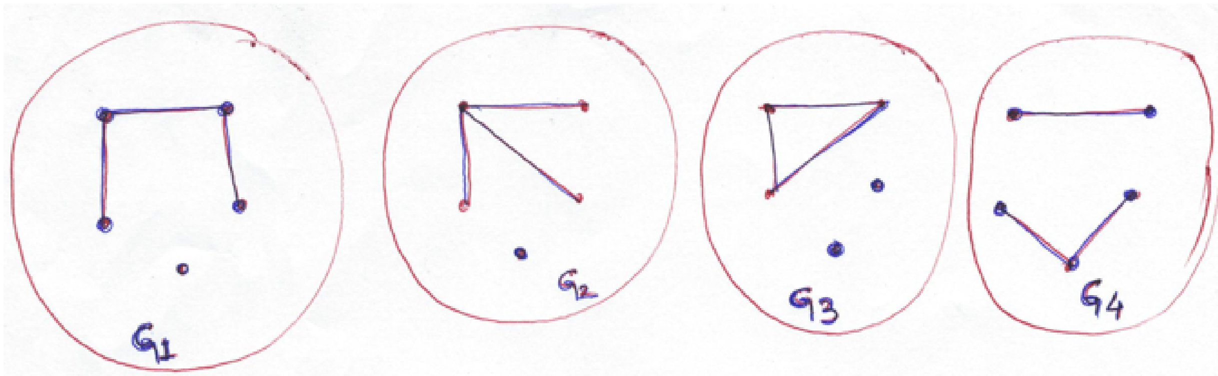
4. Construct a simple Eulerian graph on 10 vertices that contains the Petersen graph as a subgraph. [3]

   Solution.



5. Construct all the non-isomorphic graphs with 5 vertices and 3 edges. [4]

Solution. There are 4 such graphs. [4×1]



6. Let G be a connected graph with 13 vertices and 76 edges. Show that G is Hamiltonian. Is G also Eulerian? Explain. [5]

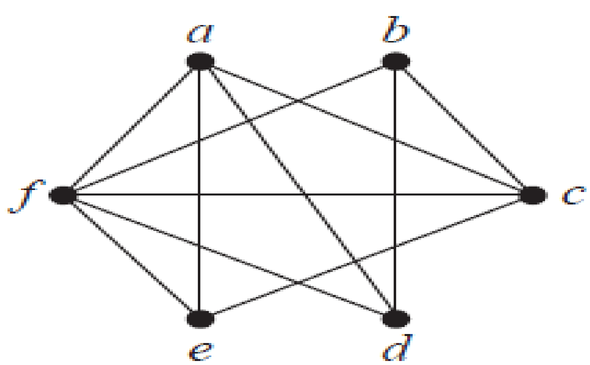
Solution. A graph on 13 vertices can have at most $\binom{13}{2} = 78$. Thus $G = K_{13} \setminus \{e_1, e_2\}$. Hence the minimum degree in G is $\delta(G) \geq 12 - 2 = 10$. [2]

Recall the Dirac's theorem, a graph with n vertices is Hamiltonian if degree of each vertex is $\geq \frac{n}{2}$.

Since $\delta(G) \geq 10 \geq \frac{13}{2} = 6.5$, G is Hamiltonian. [1]

The graph G need not be Eulerian as G can be obtained from K_{13} by deleting two edges in such a way that G has four vertices of degree 11. [2]

7. Consider the following graph G. [8]



(a) Find the radius and center of G.

Solution. Note that $e(a) = e(b) = e(c) = e(d) = e(e) = 2$ and $e(f) = 1$ [1].

Thus the radius of G is $r(G) = min\{e(v) \mid v \in V(G)\} = 1$. [1/2]

Hence the center of G is $C(G) = V(G) = \{f\}$. [1/2]

(b) Find the girth and circumference of G.

Solution. The girth of G is $g(G) = 3$, consider the cycle $C_3(a, e, f)$. [1/2+1/2]

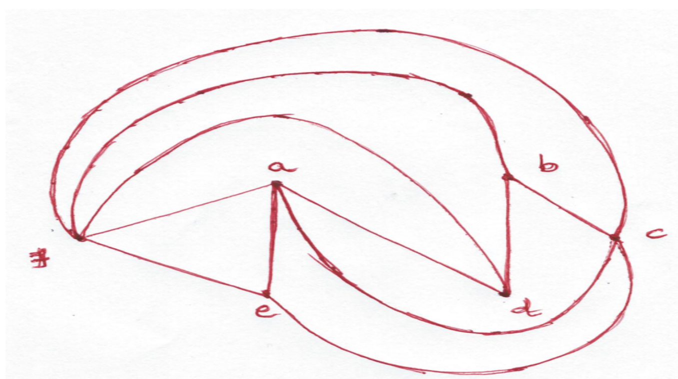
The circumference of G is 6, consider the cycle $C_6(a, c, e, f, b, d)$. [1/2+1/2]

(c) Determine whether the graph G is Eulerian, Hamiltonian, or planar. Give a proper justification.

Solution. G is not Eulerian as there is a vertex in G having odd degree, consider the vertex b. [1]

G is Hamiltonian, consider the cycle $C_6(a, c, e, f, b, d)$. [1]

G is planar, see the below planar drawing of G. [2]



8. Consider the group $U(10) = \{1, 3, 7, 9\}$ under multiplication modulo 10. [4]

(a) Find the inverse of every non-identity element.

Solution. $3^{-1} = 7$, $7^{-1} = 3$, and $9^{-1} = 9$. [1]

(b) Find the order of every non-identity element.

Solution. Since 4 is the least positive integer s.t. $3^4 = 1$ and $7^4 = 1$. Hence $o(3) = o(7) = 4$. Similarly $o(9) = 2$. [1]

(c) Is the group cyclic? If yes, write all the generators.

Solution. Yes. Since $U(10) =< 3 >=< 7 >$. Thus the generators are 3 and 7. [2]